

MAXIMAL ESTIMATES FOR SCHRÖDINGER EQUATION WITH INVERSE-SQUARE POTENTIAL

CHANGXING MIAO, JUNYONG ZHANG, AND JIQIANG ZHENG

ABSTRACT. In this paper, we consider the maximal estimates for the solution to an initial value problem of the linear Schrödinger equation with a singular potential. We show a result about the pointwise convergence of solutions to this special variable coefficient Schrödinger equation with initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s > 1/2$ or radial initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s \geq 1/4$ and the solution does not converge when $s < 1/4$.

Key Words: Inverse square potential, Maximal estimate, Spherical harmonics

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1. INTRODUCTION AND STATEMENT OF MAIN RESULT

We study the maximal estimates for the solution to an initial value problem of the linear Schrödinger equation with an inverse square potential. More precisely, we consider the following Schrödinger equation

$$(1.1) \quad \begin{cases} i\partial_t u - \Delta u + \frac{a}{|x|^2} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \ a > -(n-2)^2/4, \\ u(x, 0) = u_0(x). \end{cases}$$

The scale-covariance elliptic operator $P_a := -\Delta + \frac{a}{|x|^2}$ appearing in (1.1) plays a key role in many problems of physics and geometry. The heat and wave flows for the elliptic operator P_a have been studied in the theory of combustion (see [28]), and in the wave propagation on conic manifolds (see [8]). The Schrödinger equation (1.1) arises in the study of quantum mechanics [10]. There has been a lot of interest in developing Strichartz estimates both for the Schrödinger and wave equations with the inverse square potential, we refer the reader to Burq etc. [3, 4, 16, 17] and the authors [13]. However, as far as we known, there is few result about the maximal estimates associated with the operator P_a , which arises in the study of pointwise convergence problem for the Schrödinger and wave equations with the inverse square potential. In this paper, we aim to address some maximal estimates in the special settings associated with the operator P_a . As a direct consequence, we obtain the pointwise convergence result for $u_0 \in H^s(\mathbb{R}^n)$ with $s > 1/2$.

In the case of the free Schrödinger equation without potential, i.e. $a = 0$, there are a large amount of literature in developing the maximal estimate for its solution, which can be formally written as

$$u(t, x) = e^{it\Delta} u_0(x) = \int_{\mathbb{R}^n} e^{2\pi i(x \cdot \xi - t|\xi|^2)} \hat{u}_0(\xi) d\xi.$$

When $n = 1$, Carleson [5] proved the convergence result holds in sense of that $\lim_{t \rightarrow 0} u(t) = u_0$, a.e. x when $u_0 \in H^s(\mathbb{R})$ with $s \geq 1/4$. Dahlberg-Kenig [7] showed that the result is sharp in the sense that the solution does not converge when $s < 1/4$. When $n \geq 2$, Sjölin [22] and Vega [27] independently proved the convergence results hold when $u_0 \in H^s(\mathbb{R}^n)$ when $s > 1/2$. It follows from the construction of Dahlberg-Kenig [7], or alternatively Vega [27] that the solution does not converge when $s < 1/4$. When $n = 2$, Bourgain [1] showed that there is a certain $s < 1/2$ such that the convergence result holds, and this result was improved by Moyua-Vargas-Vega [12]. Having shown the bilinear restriction estimates for paraboloids, Tao-Vargas [25] and Tao [24] showed the convergence result holds for $s > 15/32$ and $s > 2/5$ respectively. The result was improved further to $s > 3/8$ by Lee [11] and Shao [23]. Very recently, Bourgain [2] made some progress in high dimension $n \geq 2$ to show that the convergence result holds for $s > 1/2 - 1/(4n)$ when $n \geq 1$ and the convergence result needs $s \geq (n - 2)/(2n)$ when $n \geq 5$.

In the situation when $a \neq 0$, the equation (1.1) can be viewed as a special Schrödinger equation with variable singular coefficients. The potential prevents us from using the Fourier transform to give the expression of the solution. With the motivation of regarding the potential term as a perturbation on angular direction in [3, 13, 16], we express the solution by using the Hankel transform of radial functions and spherical harmonics. Instead of Fourier transform, we utilize the Hankel transform and modify the argument of Vega [27] to show the pointwise convergence result holds when the initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s > 1/2$, or radial initial data $u_0 \in H^s(\mathbb{R}^n)$ for $s \geq 1/4$, and the solution does not converge when $s < 1/4$.

Let u be the solution to (1.1), we define the maximal function by

$$(1.2) \quad u^*(x) = \sup_{|t| > 0} |u(x, t)|.$$

Our main theorems are the following:

Theorem 1.1. *Let $\beta > 1$, $n \geq 2$ and $s > \frac{1}{2}$. Then*

$$(1.3) \quad \int_{\mathbb{R}^n} |u^*(x)|^2 \frac{dx}{(1 + |x|)^\beta} \leq C \|u_0\|_{H^s(\mathbb{R}^n)}^2.$$

As a direct consequence of Theorem 1.1, we have:

Corollary 1.1. *Let $u_0 \in H^s(\mathbb{R}^n)$ with $s > \frac{1}{2}$ and $n \geq 2$. Then*

$$(1.4) \quad \lim_{t \rightarrow 0} u(t, x) = u_0(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

Theorem 1.2. *Let B^n be the open unit ball in \mathbb{R}^n . Assume that there exists a constant C independent of u_0 such that*

$$(1.5) \quad \int_{B^n} |u^*(x)|^2 dx \leq C \|u_0\|_{H^s(\mathbb{R}^n)}^2, \quad \forall u_0(x) \in H^s(\mathbb{R}^n).$$

Then $s \geq \frac{1}{4}$.

With this in mind, Theorem 1.1 is far from being sharp. Assuming that the initial data possesses additional angular regularity, we have

Theorem 1.3. *Let B^n be the open unit ball in \mathbb{R}^n and $\epsilon > 0$. Then there exists a constant C independent of u_0 such that*

$$(1.6) \quad \int_{B^n} |u^*(x)|^2 dx \leq C \|u_0\|_{H_r^{\frac{1}{4}} H_\theta^{\frac{n-1}{2} + \epsilon}}^2,$$

where for $s, s' \geq 0$

$$H_r^s H_\theta^{s'} = \left\{ g : \|g\|_{H_r^s H_\theta^{s'}} := \|(1 - \Delta_\theta)^{\frac{s'}{2}} ((1 - \Delta)^{\frac{s}{2}} g)\|_{L_{r^{n-1}dr}^2(\mathbb{R}^+; L_\theta^2(\mathbb{S}^{n-1}))} \right\}.$$

Here Δ_θ denotes the Laplace-Beltrami operator on \mathbb{S}^{n-1} .

Remarks:

i). This result implies that the pointwise convergence of solutions to (1.1) holds for radial initial data $u_0 \in H^s(\mathbb{R}^n)$ with $s \geq 1/4$.

ii). This result is an analogue of Theorem 1.1 in [6]. We remark that the parameter ϵ in [6] should be corrected for $\epsilon > 1/2$ while not $\epsilon > 0$. Thus, we generalize and improve the result in [6] by making use of a finer result proved in [9].

Now we introduce some notations. We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some large constant C which may vary from line to line and depend on various parameters, and similarly use $A \ll B$ to denote the statement $A \leq C^{-1}B$. We employ $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. If the constant C depends on a special parameter other than the above, we shall denote it explicitly by subscripts. We briefly write $A + \epsilon$ as $A+$ or $A - \epsilon$ as $A-$ for $0 < \epsilon \ll 1$. Throughout this paper, pairs of conjugate indices are written as p, p' , where $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$.

This paper is organized as follows: In the section 2, we mainly revisit the property of the Bessel functions and the Hankel transforms associated with $-\Delta + \frac{a}{|x|^2}$. Section 3 is devoted to the proofs of the theorems.

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2. PRELIMINARY

In this section, we first list some results about the Hankel transform and the Bessel functions and then show a characterization of Sobolev norm in the Hankel transform version.

We begin with recalling the expansion formula with respect to the spherical harmonics. For more details, we refer to Stein-Weiss [21]. For the sake of convenience, let

$$(2.1) \quad \xi = \rho\omega \quad \text{and} \quad x = r\theta \quad \text{with} \quad \omega, \theta \in \mathbb{S}^{n-1}.$$

For any $g \in L^2(\mathbb{R}^n)$, the expansion formula with respect to the spherical harmonics yields

$$g(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(r) Y_{k,\ell}(\theta)$$

where

$$\{Y_{k,1}, \dots, Y_{k,d(k)}\}$$

is the orthogonal basis of the spherical harmonics space of degree k on \mathbb{S}^{n-1} , called \mathcal{H}^k , with the dimension

$$d(k) = \frac{2k+n-2}{k} C_{n+k-3}^{k-1} \simeq \langle k \rangle^{n-2}.$$

We remark that for $n = 2$, the dimension of \mathcal{H}^k is a constant, which is independent of k . Obviously, we have the orthogonal decomposition

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k.$$

By orthogonality, it gives

$$(2.2) \quad \|g(x)\|_{L^2_\theta(\mathbb{S}^{n-1})} = \|a_{k,\ell}(r)\|_{\ell^2_{k,\ell}}.$$

From $-\Delta_\theta Y_{k,\ell}(\theta) = k(k+n-2)Y_{k,\ell}(\theta)$, the fractional power of $1 - \Delta_\theta$ can be written explicitly [15]

$$(2.3) \quad (1 - \Delta_\theta)^{\frac{s}{2}} g(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1 + k(k+n-2))^{\frac{s}{2}} a_{k,\ell}(r) Y_{k,\ell}(\theta).$$

For our purpose, we need the Fourier transform of $a_{k,\ell}(r)Y_{k,\ell}(\theta)$. Theorem 3.10 in [21] asserts the Hankel transform formula

$$(2.4) \quad \hat{g}(\rho\omega) \sim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\omega) \rho^{-\frac{n-2}{2}} \int_0^\infty J_{k+\frac{n-2}{2}}(2\pi r\rho) a_{k,\ell}(r) r^{\frac{n}{2}} dr.$$

Here the Bessel function $J_k(r)$ of order k is defined by the integral

$$J_k(r) = \frac{(r/2)^k}{\Gamma(k + \frac{1}{2})\Gamma(1/2)} \int_{-1}^1 e^{isr} (1-s^2)^{(2k-1)/2} ds \quad \text{with } k > -\frac{1}{2} \text{ and } r > 0.$$

A simple computation gives the rough estimates

$$(2.5) \quad |J_k(r)| \leq \frac{Cr^k}{2^k \Gamma(k + \frac{1}{2}) \Gamma(1/2)} \left(1 + \frac{1}{k + 1/2}\right),$$

where C is a absolute constant. This estimate will be mainly used when $r \lesssim 1$. Another well known asymptotic expansion about the Bessel function is

$$(2.6) \quad J_k(r) = r^{-1/2} \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{k\pi}{2} - \frac{\pi}{4}\right) + O_k(r^{-3/2}), \quad \text{as } r \rightarrow \infty$$

but with a constant depending on k (see [21]). As pointed out in [20], if one seeks a uniform bound for large r and k , then the best one can do is $|J_k(r)| \leq Cr^{-\frac{1}{3}}$. One will find that this decay doesn't lead to the desirable result. Moreover, we recall the

properties of Bessel function $J_k(r)$ in [18, 20], we refer the readers to [14] for the detailed proof.

Lemma 2.1 (Asymptotics of the Bessel function). *Assume that $k \in \mathbb{N}$ and $k \gg 1$. Let $J_k(r)$ be the Bessel function of order k defined as above. Then there exist a large constant C and small constant c independent of k and r such that:*

- when $r \leq \frac{k}{2}$

$$(2.7) \quad |J_k(r)| \leq C e^{-c(k+r)};$$

- when $\frac{k}{2} \leq r \leq 2k$

$$(2.8) \quad |J_k(r)| \leq C k^{-\frac{1}{3}} (k^{-\frac{1}{3}} |r - k| + 1)^{-\frac{1}{4}};$$

- when $r \geq 2k$

$$(2.9) \quad J_k(r) = r^{-\frac{1}{2}} \sum_{\pm} a_{\pm}(r, k) e^{\pm i r} + E(r, k),$$

where $|a_{\pm}(r, k)| \leq C$ and $|E(r, k)| \leq C r^{-1}$.

As a consequence of Lemma 2.1, we have

Lemma 2.2. *Let $R \gg 1$. Then there exists a constant C independent of k, R such that*

$$(2.10) \quad \int_R^{2R} |J_k(r)|^2 dr \leq C.$$

Proof. To prove (2.10), we write

$$\int_R^{2R} |J_k(r)|^2 dr = \int_{I_1} |J_k(r)|^2 dr + \int_{I_2} |J_k(r)|^2 dr + \int_{I_3} |J_k(r)|^2 dr$$

where $I_1 = [R, 2R] \cap [0, \frac{k}{2}]$, $I_2 = [R, 2R] \cap [\frac{k}{2}, 2k]$ and $I_3 = [R, 2R] \cap [2k, \infty]$. By (2.7) and (2.9), we have

$$(2.11) \quad \int_{I_1} |J_k(r)|^2 dr \leq C \int_{I_1} e^{-cr} dr \leq C e^{-cR},$$

and

$$(2.12) \quad \int_{I_3} |J_k(r)|^2 dr \leq C.$$

On the other hand, one has by (2.8)

$$\int_{[\frac{k}{2}, 2k]} |J_k(r)|^2 dr \leq C \int_{[\frac{k}{2}, 2k]} k^{-\frac{2}{3}} (1 + k^{-\frac{1}{3}} |r - k|)^{-\frac{1}{2}} dr \leq C.$$

Observing $[R, 2R] \cap [\frac{k}{2}, 2k] = \emptyset$ unless $R \sim k$, we obtain

$$(2.13) \quad \int_{I_2} |J_k(r)|^2 dr \leq C.$$

This together with (2.11) and (2.12) yields (2.10). □

For simplicity, we define

$$(2.14) \quad \mu(k) = \frac{n-2}{2} + k, \quad \text{and} \quad \nu(k) = \sqrt{\mu^2(k) + a} \quad \text{with} \quad a > -(n-2)^2/4.$$

We sometime briefly write ν as $\nu(k)$. Let f be Schwartz function defined on \mathbb{R}^n , we define the Hankel transform of order ν

$$(2.15) \quad (\mathcal{H}_\nu f)(\xi) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) f(r\omega) r^{n-1} dr,$$

where $\rho = |\xi|$, $\omega = \xi/|\xi|$ and J_ν is the Bessel function of order ν . In particular, the function f is radial, then we have

$$(2.16) \quad (\mathcal{H}_\nu f)(\rho) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) f(r) r^{n-1} dr.$$

If $f(x) = \sum_{k=0}^\infty \sum_{\ell=1}^{d(k)} a_{k,\ell}(r) Y_{k,\ell}(\theta)$, it follows from (2.4) that

$$(2.17) \quad \hat{f}(\xi) = \sum_{k=0}^\infty \sum_{\ell=1}^{d(k)} 2\pi i^k Y_{k,\ell}(\omega) (\mathcal{H}_{\mu(k)} a_{k,\ell})(\rho).$$

The following properties of the Hankel transform are obtained in [3, 16]:

Lemma 2.3. *Let \mathcal{H}_ν be defined above and $A_{\nu(k)} := -\partial_r^2 - \frac{n-1}{r}\partial_r + [\nu^2(k) - (\frac{n-2}{2})^2]r^{-2}$. Then*

- (i) $\mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$,
- (ii) \mathcal{H}_ν is self-adjoint, i.e. $\mathcal{H}_\nu = \mathcal{H}_\nu^*$,
- (iii) \mathcal{H}_ν is an L^2 isometry, i.e. $\|\mathcal{H}_\nu \phi\|_{L_\xi^2} = \|\phi\|_{L_x^2}$,
- (iv) $\mathcal{H}_\nu(A_\nu \phi)(\xi) = |\xi|^2 (\mathcal{H}_\nu \phi)(\xi)$, for $\phi \in L^2$.

We next recall the following almost orthogonality inequality. Denote by P_j and \tilde{P}_j the usual dyadic frequency localization at $|\xi| \sim 2^j$ and the localization with respect to $(-\Delta + \frac{a}{|x|^2})^{\frac{1}{2}}$. We define the projectors $M_{jj'} = P_j \tilde{P}_{j'}$ and $N_{jj'} = \tilde{P}_j P_{j'}$. More precisely, let f be in the k 'th harmonic subspace, then

$$P_j f = \mathcal{H}_{\mu(k)} \beta_j \mathcal{H}_{\mu(k)} f \quad \text{and} \quad \tilde{P}_j f = \mathcal{H}_{\nu(k)} \beta_j \mathcal{H}_{\nu(k)} f,$$

where $\beta_j(\xi) = \beta(2^{-j}|\xi|)$ with $\beta \in C_0^\infty(\mathbb{R}^+)$ supported in $[\frac{1}{2}, 2]$. Then we have the following almost orthogonality inequality [3]:

Lemma 2.4 (Almost orthogonality inequality). *Let $f \in L^2(\mathbb{R}^n)$, then there exists a constant C independent of j, j' such that*

$$(2.18) \quad \|M_{jj'} f\|_{L^2(\mathbb{R}^n)}, \|N_{jj'} f\|_{L^2(\mathbb{R}^n)} \leq C 2^{-\epsilon|j-j'|} \|f\|_{L^2(\mathbb{R}^n)},$$

where $\epsilon < 1 + \min\{\frac{n-2}{2}, (\frac{(n-2)^2}{4} + a)^{\frac{1}{2}}\}$.

As a consequence, we have

Lemma 2.5. *Let $f \in L^2(\mathbb{R}^n)$ such that $f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(r) Y_{k,\ell}(\theta)$. Then for $0 \leq s < 1 + \min\{\frac{n-2}{2}, (\frac{(n-2)^2}{4} + a)^{\frac{1}{2}}\}$ and $s' \geq 0$*

$$(2.19) \quad \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} M^{2s} (1+k)^{2s'} \|b_{k,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L_{\rho}^2}^2 \sim \|f\|_{\dot{H}_r^s H_{\theta}^{s'}}^2,$$

where $b_{k,\ell}(\rho) = (\mathcal{H}_{\nu(k)} a_{k,\ell})(\rho)$ and $\chi \in C_0^{\infty}(\mathbb{R}^n)$ such that $\text{supp } \chi \subset [1/2, 1]$.

Proof. Note that $-\Delta_{\theta} Y_{k,\ell} = k(k+n-2) Y_{k,\ell}$, then we have by Lemma 2.3

$$\begin{aligned} \|f\|_{\dot{H}_r^0 H_{\theta}^{s'}}^2 &\sim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1+k)^{2s'} \|a_{k,\ell}(r)\|_{L_{r^{n-1}dr}^2(\mathbb{R}^+)}^2 \|Y_{k,\ell}(\theta)\|_{L_{\theta}^2}^2 \\ &\sim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1+k)^{2s'} \|b_{k,\ell}(\rho)\|_{L_{\rho^{n-1}d\rho}^2(\mathbb{R}^+)}^2. \end{aligned}$$

By (2.3), it suffices to show (2.19) with $s' = 0$. By Lemma 2.3, we have

$$\begin{aligned} \|b_{k,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L_{\rho}^2} &= \|\chi(\frac{\rho}{M}) \mathcal{H}_{\nu} [Y_{k,\ell}(\theta) a_{k,\ell}(r)](\xi)\|_{L_{\xi}^2} \\ &= \|\mathcal{H}_{\nu} [\chi(\frac{\rho}{M}) \mathcal{H}_{\nu} (Y_{k,\ell}(\theta) a_{k,\ell}(r))(\xi)]\|_{L_x^2}. \end{aligned}$$

This yields that by letting let $j = \log_2 M$

$$\begin{aligned} \|b_{k,\ell}(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L_{\rho}^2} &= \|[\mathcal{H}_{\nu} \chi(\frac{\rho}{M}) \mathcal{H}_{\nu}] (Y_{k,\ell}(\theta) a_{k,\ell}(r))\|_{L_x^2} \\ &= \|\tilde{P}_j (Y_{k,\ell}(\theta) a_{k,\ell}(r))\|_{L_x^2}. \end{aligned}$$

Let $g_{k,\ell}(x) = Y_{k,\ell}(\theta) a_{k,\ell}(r)$ and $\overline{P_{j'}} = P_{j'-1} + P_{j'} + P_{j'+1}$. We have by the triangle inequality and Lemma 2.4

$$\begin{aligned} \text{L.H.S of (2.19)} &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{j \in \mathbb{Z}} 2^{2sj} \|\tilde{P}_j g_{k,\ell}\|_{L_x^2}^2 \\ &\lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{j \in \mathbb{Z}} 2^{2sj} \left(\sum_{j'} \|\tilde{P}_j \overline{P_{j'}} P_{j'} g_{k,\ell}\|_{L_x^2} \right)^2 \\ &\lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{j \in \mathbb{Z}} 2^{2sj} \left(\sum_{j'} 2^{-\epsilon|j-j'|} \|P_{j'} g_{k,\ell}\|_{L_x^2} \right)^2, \end{aligned}$$

where $s < \epsilon < 1 + \min\{\frac{n-2}{2}, (\frac{(n-2)^2}{4} + a)^{\frac{1}{2}}\}$. Let $0 < \epsilon_1 \ll 1$ such that $\epsilon_2 := \epsilon - \epsilon_1 > s$, then

$$\begin{aligned} \text{L.H.S of (2.19)} &\leq C \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{j'} 2^{-2\epsilon_2|j-j'|} \|P_{j'} g_{k,\ell}\|_{L^2(\mathbb{R}^n)}^2 \sum_{j'} 2^{-2\epsilon_1|j-j'|} \\ &\leq C \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{j'} 2^{2j's} \sum_{j \in \mathbb{Z}} 2^{2js} 2^{-2\epsilon_2|j|} \|P_{j'} g_{k,\ell}\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq C \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{j'} 2^{2j's} \|P_{j'} g_{k,\ell}\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

By the definition of $P_{j'}$, Lemma 2.3 and (2.17), we have

$$\begin{aligned} \text{L.H.S of (2.19)} &\leq C \sum_{j'} 2^{2j's} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| \chi\left(\frac{\rho}{2^{j'}}\right) [\mathcal{H}_{\mu(k)} a_{k,\ell}](\rho) \rho^{\frac{n-1}{2}} \right\|_{L^2(\mathbb{R}^+)}^2 \\ &= C \sum_{j'} 2^{2j's} \left\| \chi\left(\frac{\rho}{2^{j'}}\right) \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} 2\pi i^k [\mathcal{H}_{\mu(k)} a_{k,\ell}](\rho) Y_{k,\ell}(\omega) \right\|_{L^2(\mathbb{R}^n)}^2 \\ &= C \sum_{j'} 2^{2j's} \left\| \chi\left(\frac{\rho}{2^{j'}}\right) \hat{f} \right\|_{L^2(\mathbb{R}^n)}^2 \sim \|f\|_{\dot{H}^s}^2. \end{aligned}$$

We can use the similar argument to prove

$$\text{L.H.S of (2.19)} \geq c \|f\|_{\dot{H}^s}^2.$$

Therefore we conclude the proof of Lemma 2.4. \square

3. PROOF OF THE MAIN THEOREMS

In this section, we first use the spherical harmonic expansion to write the solution as a linear combination of products of the Hankel transform of radial functions and spherical harmonics. We prove the main theorems by analyzing the property of the Hankel transform. The key ingredients are to use the stationary phase argument and to exploit the asymptotics behavior of the Bessel function.

3.1. The expression of the solution. Consider the following Cauchy problem:

$$(3.1) \quad \begin{cases} i\partial_t u - \Delta u + \frac{a}{|x|^2} u = 0, \\ u(x, 0) = u_0(x). \end{cases}$$

We use the spherical harmonic expansion to write

$$(3.2) \quad u_0(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}^0(r) Y_{k,\ell}(\theta).$$

Let us consider the equation (3.1) in polar coordinates. Write $v(t, r, \theta) = u(t, r\theta)$ and $g(r, \theta) = u_0(r\theta)$. Then $v(t, r, \theta)$ satisfies that

$$(3.3) \quad \begin{cases} i\partial_t v - \partial_{rr} v - \frac{n-1}{r}\partial_r v - \frac{1}{r^2}\Delta_\theta v + \frac{a}{r^2}v = 0 \\ v(0, r, \theta) = g(r, \theta). \end{cases}$$

By (3.2), we have

$$g(r, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}^0(r) Y_{k,\ell}(\theta).$$

Using separation of variables, we can write v as a linear combination of products of radial functions and spherical harmonics

$$(3.4) \quad v(t, r, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} v_{k,\ell}(t, r) Y_{k,\ell}(\theta),$$

where $v_{k,\ell}$ is given by

$$\begin{cases} i\partial_t v_{k,\ell} - \partial_{rr} v_{k,\ell} - \frac{n-1}{r}\partial_r v_{k,\ell} + \frac{k(k+n-2)+a}{r^2} v_{k,\ell} = 0, \\ v_{k,\ell}(0, r) = a_{k,\ell}^0(r) \end{cases}$$

for each $k, \ell \in \mathbb{N}$, $1 \leq \ell \leq d(k)$. Then it reduces to consider by the definition of $A_{\nu(k)}$

$$(3.5) \quad \begin{cases} i\partial_t v_{k,\ell} + A_{\nu(k)} v_{k,\ell} = 0, \\ v_{k,\ell}(0, r) = a_{k,\ell}^0(r). \end{cases}$$

Applying the Hankel transform to the equation (3.5), by (iv) in Lemma 2.3, we have

$$(3.6) \quad \begin{cases} i\partial_t \tilde{v}_{k,\ell} + \rho^2 \tilde{v}_{k,\ell} = 0 \\ \tilde{v}_{k,\ell}(0, \rho) = b_{k,\ell}^0(\rho), \end{cases}$$

where

$$(3.7) \quad \tilde{v}_{k,\ell}(t, \rho) = (\mathcal{H}_\nu v_{k,\ell})(t, \rho), \quad b_{k,\ell}^0(\rho) = (\mathcal{H}_\nu a_{k,\ell}^0)(\rho).$$

Solving this ODE and inverting the Hankel transform, we obtain

$$\begin{aligned} v_{k,\ell}(t, r) &= \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) \tilde{v}_{k,\ell}(t, \rho) \rho^{n-1} d\rho \\ &= \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) e^{it\rho^2} b_{k,\ell}^0(\rho) \rho^{n-1} d\rho. \end{aligned}$$

Therefore we get

$$(3.8) \quad \begin{aligned} u(x, t) &= v(t, r, \theta) \\ &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) e^{it\rho^2} b_{k,\ell}^0(\rho) \rho^{n-1} d\rho \\ &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \mathcal{H}_{\nu(k)}[e^{it\rho^2} b_{k,\ell}^0(\rho)](r). \end{aligned}$$

3.2. Proof of the Theorem 1.1. In this subsection, we prove the Theorem 1.1. By the Sobolev embedding $\dot{H}^{\frac{1}{2}-}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, it suffices to show

Proposition 3.1. *Let $\alpha \geq \frac{1}{2} - \frac{\beta}{4}$ and $\beta = 1+$ such that*

$$2\alpha - 1 + \frac{\beta}{2} < 1 + \min \left\{ (n-2)/2, ((n-2)^2/4 + a)^{\frac{1}{2}} \right\},$$

then there exists a constant C independent of u_0 such that

$$(3.9) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\partial_t^\alpha u(x, t)|^2 \frac{dt dx}{(1+|x|)^\beta} \leq C \|u_0\|_{\dot{H}^{2\alpha-1+\frac{\beta}{2}}(\mathbb{R}^n)}^2.$$

Proof. By the Plancherel theorem with respect to time t , we obtain

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} |\partial_t^\alpha u(x, t)|^2 \frac{dt dx}{(1+|x|)^\beta} = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \left| \tau^\alpha \int_{\mathbb{R}} e^{-it\tau} u(x, t) dt \right|^2 \frac{d\tau dx}{(1+|x|)^\beta}.$$

Using (3.8), we further have

$$\begin{aligned} & \text{L.H.S of (3.9)} \\ & \lesssim \int_{\mathbb{R}^{n+1}} \left| \tau^\alpha \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \int_{\mathbb{R}} \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) e^{it(\rho^2-\tau)} b_{k,\ell}^0(\rho) \rho^{n-1} d\rho dt \right|^2 \frac{d\tau dx}{(1+|x|)^\beta} \\ & \lesssim \int_{\mathbb{R}^{n+1}} \left| \tau^\alpha \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) b_{k,\ell}^0(\rho) \rho^{n-1} \delta(\tau - \rho^2) d\rho \right|^2 \frac{d\tau dx}{(1+|x|)^\beta} \\ & \lesssim \int_{\mathbb{R}^n} \int_0^\infty \left| \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \rho^\alpha (r\sqrt{\rho})^{-\frac{n-2}{2}} J_{\nu(k)}(r\sqrt{\rho}) b_{k,\ell}^0(\sqrt{\rho}) \rho^{\frac{n-1}{2}} \rho^{-\frac{1}{2}} \right|^2 \frac{d\rho dx}{(1+|x|)^\beta}. \end{aligned}$$

By the orthogonality, we see that

$$(3.10) \quad \begin{aligned} & \text{L.H.S of (3.9)} \\ & \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \int_0^\infty \int_0^\infty \left| \rho^{2\alpha+\frac{1}{2}} (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) b_{k,\ell}^0(\rho) \rho^{n-2} \right|^2 \frac{d\rho r^{n-1} dr}{(1+r)^\beta}. \end{aligned}$$

Let χ be a smoothing function supported in $[1, 2]$. For our purpose, we make a dyadic decomposition to obtain

$$\begin{aligned} & \text{L.H.S of (3.9)} \\ & \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} \int_0^\infty \int_0^\infty \left| \rho^{2\alpha+\frac{1}{2}} (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) b_{k,\ell}^0(\rho) \rho^{n-2} \chi\left(\frac{\rho}{M}\right) \right|^2 \frac{r^{n-1} dr d\rho}{(1+r)^\beta} \\ & \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} M^{2(n-2+2\alpha+\frac{1}{2})+1-n} \int_0^\infty \int_0^\infty \left| (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) b_{k,\ell}^0(M\rho) \chi(\rho) \right|^2 \frac{r^{n-1} dr d\rho}{(1+\frac{r}{M})^\beta} \\ & \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} \sum_{R \in 2^{\mathbb{Z}}} M^{n-2+4\alpha} R^{n-1} \int_R^{2R} \int_0^\infty \left| (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) b_{k,\ell}^0(M\rho) \chi(\rho) \right|^2 \frac{dr d\rho}{(1+\frac{r}{M})^\beta}. \end{aligned}$$

Define

$$(3.11) \quad G_{k,\ell}(R, M) = \int_R^{2R} \int_0^\infty |(r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) b_{k,\ell}^0(M\rho) \chi(\rho)|^2 \frac{dr d\rho}{(1 + \frac{r}{M})^\beta}.$$

Proposition 3.2. *We have the following inequality*

$$(3.12) \quad G_{k,\ell}(R, M) \lesssim \begin{cases} R^{2\nu(k)-n+3} M^{-n} \min \left\{ 1, \left(\frac{M}{R} \right)^\beta \right\} \|b_{k,\ell}^0(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2, & R \lesssim 1; \\ \min \left\{ 1, \left(\frac{M}{R} \right)^\beta \right\} R^{-(n-2)} M^{-n} \|b_{k,\ell}^0(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2, & R \gg 1. \end{cases}$$

Proof. To prove (3.12), we break it into two cases.

• Case 1: $R \lesssim 1$. Since $\rho \sim 1$, we have $r\rho \lesssim 1$. By the property of the Bessel function (2.5), we obtain

$$(3.13) \quad \begin{aligned} G_{k,\ell}(R, M) &\lesssim \int_R^{2R} \int_0^\infty \left| \frac{(r\rho)^{\nu(k)} (r\rho)^{-\frac{n-2}{2}}}{2^{\nu(k)} \Gamma(\nu(k) + \frac{1}{2}) \Gamma(\frac{1}{2})} b_{k,\ell}^0(M\rho) \chi(\rho) \right|^2 d\rho \frac{dr}{(1 + \frac{r}{M})^\beta} \\ &\lesssim R^{2\nu(k)-n+3} M^{-n} \min \left\{ 1, \left(\frac{M}{R} \right)^\beta \right\} \|b_{k,\ell}^0(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2. \end{aligned}$$

• Case 2: $R \gg 1$. Since $\rho \sim 1$, we have $r\rho \gg 1$. We estimate

$$(3.14) \quad G_{k,\ell}(R, M) \lesssim R^{-(n-2)} \int_0^\infty |b_{k,\ell}^0(M\rho) \chi(\rho)|^2 \int_R^{2R} |J_{\nu(k)}(r\rho)|^2 \frac{dr}{(1 + \frac{r}{M})^\beta} d\rho.$$

(i) Subcase: $R \lesssim M$. Noting that $\rho \sim 1$, we obtain by Lemma 2.2

$$(3.15) \quad \int_R^{2R} |J_{\nu(k)}(r\rho)|^2 \frac{dr}{(1 + \frac{r}{M})^\beta} \lesssim \int_R^{2R} |J_{\nu(k)}(r\rho)|^2 dr \lesssim 1.$$

(ii) Subcase: $R \gg M$. Noticing that $\rho \sim 1$ again, we obtain by Lemma 2.2

$$(3.16) \quad \int_R^{2R} |J_{\nu(k)}(r\rho)|^2 \frac{dr}{(1 + \frac{r}{M})^\beta} \lesssim \left(\frac{M}{R} \right)^\beta \int_R^{2R} |J_{\nu(k)}(r\rho)|^2 dr \lesssim \left(\frac{M}{R} \right)^\beta.$$

Putting (3.15) and (3.16) into (3.14), we have

$$\begin{aligned} G_{k,\ell}(R, M) &\lesssim \min \left\{ 1, \left(\frac{M}{R} \right)^\beta \right\} R^{-(n-2)} \int_0^\infty |b_{k,\ell}^0(M\rho) \chi(\rho)|^2 d\rho \\ &\lesssim \min \left\{ 1, \left(\frac{M}{R} \right)^\beta \right\} R^{-(n-2)} M^{-n} \|b_{k,\ell}^0(\rho) \chi(\frac{\rho}{M}) \rho^{\frac{n-1}{2}}\|_{L^2}^2. \end{aligned}$$

Thus we prove (3.12). □

Now we return to prove Proposition 3.1. By Proposition 3.2, we show

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\partial_t^\alpha u(x, t)|^2 \frac{dt dx}{(1 + |x|)^\beta} \\ & \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} \sum_{\{R \in 2^{\mathbb{Z}}: R \lesssim 1\}} M^{4\alpha-2} R^{2(\nu(k)+1)} \min \left\{ 1, \left(\frac{M}{R} \right)^\beta \right\} \|b_{k,\ell}^0(\rho) \chi\left(\frac{\rho}{M}\right) \rho^{\frac{n-1}{2}}\|_{L^2}^2 \\ & + \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} \sum_{\{R \in 2^{\mathbb{Z}}: R \gg 1\}} M^{4\alpha-2+\beta} R^{1-\beta} \|b_{k,\ell}^0(\rho) \chi\left(\frac{\rho}{M}\right) \rho^{\frac{n-1}{2}}\|_{L^2}^2. \end{aligned}$$

From $\beta = 1+$, one has

$$\begin{aligned} & \sum_{M \in 2^{\mathbb{Z}}} \sum_{\{R \in 2^{\mathbb{Z}}: R \lesssim 1\}} M^{4\alpha-2} R^{2(\nu(k)+1)} \min \left\{ 1, \left(\frac{M}{R} \right)^\beta \right\} \|b_{k,\ell}^0(\rho) \chi\left(\frac{\rho}{M}\right) \rho^{\frac{n-1}{2}}\|_{L^2}^2 \\ & \lesssim \sum_{M \in 2^{\mathbb{Z}}} M^{4\alpha-2+\beta} \|b_{k,\ell}^0(\rho) \chi\left(\frac{\rho}{M}\right) \rho^{\frac{n-1}{2}}\|_{L^2}^2. \end{aligned}$$

Since $\alpha \geq \frac{1}{2} - \frac{\beta}{4}$, we have by Lemma 2.5

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\partial_t^\alpha u(x, t)|^2 \frac{dt dx}{(1 + |x|)^\beta} & \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \sum_{M \in 2^{\mathbb{Z}}} M^{4\alpha-2+\beta} \|b_{k,\ell}^0(\rho) \chi\left(\frac{\rho}{M}\right) \rho^{\frac{n-1}{2}}\|_{L^2}^2 \\ & \leq C \|u_0\|_{\dot{H}^{2\alpha-1+\frac{\beta}{2}}(\mathbb{R}^n)}^2. \end{aligned}$$

□

Finally, we apply Proposition 3.1 with $\alpha = \frac{1}{2}+$ and $\alpha = \frac{1}{2}-$ to prove Theorem 1.1.

3.3. Proof of Theorem 1.2. In this subsection, we construct an example to show Theorem 1.2. The main idea is the stationary phase argument. By (3.8), we recall

$$(3.17) \quad u(x, t) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) e^{it\rho^2} b_{k,\ell}^0(\rho) \rho^{n-1} d\rho,$$

where

$$b_{k,\ell}^0(\rho) = (\mathcal{H}_\nu a_{k,\ell}^0)(\rho), \quad u_0(x) = u_0(r\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}^0(r) Y_{k,\ell}(\theta).$$

In particular we choose $u_0(x)$ to be a radial function such that $(\mathcal{H}_{\nu(0)} u_0)(\xi) = \chi_N(|\xi|)$ where χ_N is a smooth positive function supported in J_N (to be chosen later) and $N \gg 1$. Then

$$(3.18) \quad u(x, t) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(0)}(r\rho) e^{it\rho^2} \chi_N(\rho) \rho^{n-1} d\rho.$$

Recalling the asymptotic expansion about the Bessel function

$$J_\nu(r) = r^{-1/2} \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O_\nu(r^{-3/2}), \quad \text{as } r \rightarrow \infty$$

with a constant depending on ν (see [21]), then we can write

$$(3.19) \quad \begin{aligned} u(x, t) = & C_\nu \int_0^\infty (r\rho)^{-\frac{n-1}{2}} \left(e^{i(r\rho - \frac{\nu\pi}{2} - \frac{\pi}{4})} - e^{-i(r\rho - \frac{\nu\pi}{2} - \frac{\pi}{4})} \right) e^{it\rho^2} \chi_N(\rho) \rho^{n-1} d\rho \\ & + C_\nu \int_0^\infty (r\rho)^{-\frac{n-2}{2}} O_\nu((r\rho)^{-\frac{3}{2}}) e^{it\rho^2} \chi_N(\rho) \rho^{n-1} d\rho. \end{aligned}$$

Let us define

$$(3.20) \quad I_1(r) = C_\nu e^{i(\frac{\nu\pi}{2} + \frac{\pi}{4})} \int_0^\infty (r\rho)^{-\frac{n-1}{2}} e^{i(-r\rho + t\rho^2)} \chi_N(\rho) \rho^{n-1} d\rho,$$

$$(3.21) \quad I_2(r) = C_\nu e^{-i(\frac{\nu\pi}{2} + \frac{\pi}{4})} \int_0^\infty (r\rho)^{-\frac{n-1}{2}} e^{i(r\rho + t\rho^2)} \chi_N(\rho) \rho^{n-1} d\rho,$$

and

$$(3.22) \quad I_3(r) = C_\nu \int_0^\infty (r\rho)^{-\frac{n-2}{2}} O_\nu((r\rho)^{-\frac{3}{2}}) e^{it\rho^2} \chi_N(\rho) \rho^{n-1} d\rho.$$

Let $\phi_r(\rho) = t\rho^2 - r\rho$. The fundamental idea is to choose sets J_N and $E \subset B^n$, in which $t(r)$ can be chosen, so that $\partial_\rho \phi_r(\rho) = 2t(r)\rho - r$ almost vanishes for all $\rho \in J_N$ and $r \in \{|x| : x \in E\}$. To this end, we choose

$$E = \{x : \frac{1}{100} \leq |x| \leq \frac{1}{8}\} \text{ and } J_N = [N, N + 2N^{\frac{1}{2}}].$$

Choose $t(r) = \frac{r}{2(N + \sqrt{N})}$, then $\partial_\rho \phi_r(N + N^{\frac{1}{2}}) = 0$. Thus

$$(3.23) \quad I_1(r) = C_\nu e^{i(\frac{\nu\pi}{2} + \frac{\pi}{4})} e^{i\phi_r(N + \sqrt{N})} \int_0^\infty (r\rho)^{-\frac{n-1}{2}} e^{\frac{ir[\rho - (N + \sqrt{N})]^2}{2(N + \sqrt{N})}} \chi_N(\rho) \rho^{n-1} d\rho.$$

Observe that

$$(3.24) \quad |I_1(r)| \geq c_\nu \int_0^\infty (r\rho)^{-\frac{n-1}{2}} \cos\left(\frac{r[\rho - (N + \sqrt{N})]^2}{2(N + \sqrt{N})}\right) \chi_N(\rho) \rho^{n-1} d\rho.$$

Moreover, there exists a small constant $c > 0$ such that

$$\cos\left(\frac{r[\rho - (N + \sqrt{N})]^2}{2(N + \sqrt{N})}\right) \geq c,$$

since $|\frac{r[\rho - (N + \sqrt{N})]^2}{2(N + \sqrt{N})}| \leq \frac{\pi}{4}$ for all $\rho \in J_N$ with $N \gg 1$ and $r \in [\frac{1}{100}, \frac{1}{8}]$. Therefore,

$$(3.25) \quad |I_1(r)| \geq c_\nu r^{-\frac{n-1}{2}} \int_0^\infty \chi_N(\rho) \rho^{\frac{n-1}{2}} d\rho \geq c_\nu r^{-\frac{n-1}{2}} N^{\frac{n}{2}}.$$

On the other hand, let $\varphi_r(\rho) = t\rho^2 + r\rho$, $t = t(r)$ as before, then $\partial_\rho \varphi_r(\rho) = 2t(r)\rho + r \geq \frac{1}{200}$ when $\rho \in J_N$ and $r \in [\frac{1}{100}, \frac{1}{8}]$. From the integral by parts, we obtain

$$(3.26) \quad |I_2(r)| \leq C_\nu r^{-\frac{n}{2}} N^{\frac{n-2}{2}}.$$

Obviously, we have

$$(3.27) \quad |I_3(r)| \leq C_\nu r^{-\frac{n}{2}} N^{\frac{n-2}{2}}.$$

Combining (3.25)-(3.27), we get for $N \gg 1$ and $r \in [\frac{1}{100}, \frac{1}{8}]$

$$(3.28) \quad u^*(x) \geq cN^{\frac{n}{2}}.$$

On the other hand, let $j_0 = \log_2 N$, then we obtain by the definition of P_j and \tilde{P}_j

$$\|u_0(x)\|_{H^s}^2 = \sum_j 2^{2js} \|P_j u_0\|_{L^2}^2 = \sum_j 2^{2js} \|P_j \tilde{P}_{j_0} u_0\|_{L^2}^2.$$

By Lemma 2.4, we choose $s < \epsilon < 1 + \min\{\frac{n-2}{2}, (\frac{(n-2)^2}{4} + a)^{\frac{1}{2}}\}$ to obtain

$$(3.29) \quad \begin{aligned} \|u_0(x)\|_{H^s}^2 &\leq C \sum_j 2^{2js-2\epsilon|j-j_0|} \|u_0\|_{L^2}^2 \\ &= CN^{2s} \sum_j 2^{2js-2\epsilon|j|} \|\chi_N\|_{L^2}^2 = N^{2s+n-\frac{1}{2}}. \end{aligned}$$

Thus, by (1.5) and (3.28), we must have $s \geq 1/4$.

3.4. Proof of the Theorem 1.3. In this subsection, we show Theorem 1.3. Even though there is a loss of the angular regularity in Theorem 1.3, the result implies the sharp result for the radial initial data. The key ingredient here is the following lemma proved in [9]:

Lemma 3.1. *Let $\tilde{J}_\nu(s) = s^{\frac{1}{2}} J_\nu(s)$ with $s \geq 0$, and let*

$$(3.30) \quad T_\nu g(r) = \int_I \frac{e^{it(r)\rho^2} \tilde{J}_\nu(r\rho)}{\rho^{\frac{1}{4}}} g(\rho) d\rho.$$

Then

$$(3.31) \quad \int_0^1 |T_\nu g(r)|^2 dr \leq C \int_I |g(\rho)|^2 d\rho,$$

where the constant C is independent of $g \in L^2(I)$, of the interval I , of the measurable function $t(r)$ and of the order $\nu \geq 0$.

We also can follow the Carleson approach [5] to linearize our maximal operator, by making t into a function of r , $t(r)$. By the triangle inequality, we estimate

$$\|u^*(x)\|_{L^2(B^n)} \leq C \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| \int_0^{\infty} (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) e^{it(r)\rho^2} b_{k,\ell}^0(\rho) \rho^{n-1} d\rho \right\|_{L^2_{r^{n-1}dr}}.$$

Let $g(\rho) = b_{k,\ell}^0(\rho) \rho^{\frac{n-1}{2} + \frac{1}{4}}$, then

$$(3.32) \quad \|u^*(x)\|_{L^2(B^n)} \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| \int_0^{\infty} \tilde{J}_{\nu(k)}(r\rho) e^{it(r)\rho^2} \rho^{-\frac{1}{4}} g(\rho) d\rho \right\|_{L^2_r([0,1])}.$$

Using Lemma 3.1, we obtain

$$(3.33) \quad \|u^*(x)\|_{L^2(B^n)} \lesssim C \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| b_{k,\ell}^0(\rho) \rho^{\frac{n-1}{2} + \frac{1}{4}} \right\|_{L^2_\rho(\mathbb{R}^+)}.$$

Let $\alpha = (n - 1)/2 + \epsilon$ with $\epsilon > 0$, we have by the Cauchy-Schwarz inequality

$$\|u^*(x)\|_{L^2(B^n)} \leq C \left(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1+k)^{-2\alpha} \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1+k)^{2\alpha} \|b_{k,\ell}^0(\rho) \rho^{\frac{n-1}{2} + \frac{1}{4}}\|_{L^2_\rho(\mathbb{R}^+)}^2 \right)^{\frac{1}{2}}.$$

Since $d(k) \simeq \langle k \rangle^{n-2}$, we have by Lemma 2.5

$$\|u^*(x)\|_{L^2(B^n)} \lesssim \left(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1+k)^{2\alpha} \|b_{k,\ell}^0(\rho) \rho^{\frac{n-1}{2} + \frac{1}{4}}\|_{L^2_\rho(\mathbb{R}^+)}^2 \right)^{\frac{1}{2}} \lesssim \|u_0\|_{H_r^{\frac{1}{4}} H_\theta^\alpha}.$$

This completes the proof of Theorem 1.3.

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INSTITUTE OF APPLIED PHYSICS AND COMPUTATIONAL MATHEMATICS, P. O. Box 8009, BEIJING, CHINA, 100088

E-mail address: miao_changxing@iapcm.ac.cn

DEPARTMENT OF MATHEMATICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING 100081, CHINA

E-mail address: zhang_junyong@bit.edu.cn

THE GRADUATE SCHOOL OF CHINA ACADEMY OF ENGINEERING PHYSICS, P. O. Box 2101, BEIJING, CHINA, 100088

E-mail address: zhengjiqiang@gmail.com